Bode Gain-Phase Relationship (Bode, 1945)

- Equation (2.11) in Skogestad's book is an approximation at local frequency ω_o by using Bode's gain-phase relationship, which is as follows and shown in H. W. Bode's book *Network Analysis and Feedback Design of Amplifiers*, Chapter 14.

- Bode's gain-phase relation (for a minimum phase system),

$$\angle G(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \underbrace{\frac{\mathrm{d}(\ln|G(j\omega)|)}{\mathrm{d}(\ln\omega)}}_{N(\omega)} \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right| \frac{\mathrm{d}\omega}{\omega}, \tag{1}$$

where $N(\omega_0) = \left.\frac{\mathrm{d}(\ln|G(j\omega)|)}{\mathrm{d}(\ln\omega)}\right|_{\omega=\omega_o}$ is the local slope. If we know the local slope at gain crossover frequency is constant (e.g. -40dB per decade), and further make use of the fact[1] that $\int_{-\infty}^{+\infty} \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right| \frac{\mathrm{d}\omega}{\omega} = \frac{\pi^2}{2}$, then Equation (1) can be approximated by

$$\angle G(j\omega_0) \approx \frac{\pi}{2} N(\omega_0) \text{ [in radians]}.$$

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- Proof of Equation (1) as explained in H. W. Bode's book.

Theorem 1. Bode's gain-phase relationship (1945). *Equation* (1) *holds for minimum phase systems.*

- *Proof.* Starting from Chapter 8, Bode used the notation for a complex valued function $\theta = A + iB$ where A is the real component of θ and an

[1] Bode pointed this fact to Bierenes de Haan, *Nouvelles Tables D'Intégrales Définies*, Table 256, p. 377, equations 10) and 11). See also "How to compute $\int_0^\infty \ln(1+e^{-x})\,\mathrm{d}x$?" on `math.stackexchange.com`.

even function; B is the imaginary part and an odd function due to the fact that one can write $\theta(\omega) = |\theta(\omega)|\, e^{i\angle\theta(\omega)}$. By Euler formula,

$$\begin{aligned}\theta(\omega) &= |\theta(\omega)|\, e^{i\angle\theta(\omega)}\\ &= \underbrace{|\theta(\omega)| \cos\left(\angle\theta(\omega)\right)}_{A,\ \text{even function}} + i\, \underbrace{|\theta(\omega)| \sin\left(\angle\theta(\omega)\right)}_{B,\ \text{odd function}}.\end{aligned}$$

Then Bode investigated the value of B at $\omega = \omega_c$. In order to calculate the value, first divide $\theta(\omega)$ by $\omega - \omega_c$. Also introduce the complementary point $-\omega_c$ to preserve symmetry (why?). For convenience, only $\theta(\omega) - A_c$ is considered in the integrand, where A_c is the value assumed by A at frequency ω_c. So Bode constructed a contour integral as below,

$$\int_\gamma \left(\frac{\theta(\omega) - A_c}{\omega - \omega_c} - \frac{\theta(\omega) - A_c}{\omega + \omega_c} \right) \mathrm{d}\omega = 0, \tag{2}$$

where γ is the usual Nyquist contour consisting of imaginary axis (with small semicircle indents around $\pm\omega_c$) and indefinitely large semicircle which together surround the open right half plane (ORHP). The integral of the left hand side of Equation (2) is zero because the integrand is analytic within the region formed by contour γ.

At high frequencies, the integrand goes to zero faster than ${}^1\!/\!{}_{\omega^2}$ so at infinity the integral along the indefinitely large semicircle is zero (why?). The remaining part of the integral is taken along the imaginary axis except that at two frequencies $\pm\omega_c$ two semicircle indents are used. So the contour integral in Equation (2) could be further broken up into three terms,

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{2\omega_c}{\omega^2 - \omega_c^2}(A - A_c + iB)\, \mathrm{d}\omega - \int_{\gamma_1} iB_c \left(\frac{1}{\omega - \omega_c} - \frac{1}{\omega + \omega_c} \right) \mathrm{d}\omega \\ + \int_{\gamma_2} iB_c \left(\frac{1}{\omega - \omega_c} - \frac{1}{\omega + \omega_c} \right) \mathrm{d}\omega = 0,\end{aligned} \tag{3}$$

where the first integral is taken along the imaginary axis and the latter two along two indentations γ_1 (centered at $\omega = -\omega_c$) and γ_2 (centered at $\omega = \omega_c$). Since B is an odd function, B term in the first integral in Equation (3) can be ignored. Also we can restrict the integral limits to the positive side $[0, +\infty]$ and double this integral to get integration over $[-\infty, +\infty]$ since A is an even function.

In the second integral, when we calculate integral along the indentation γ_1, $\dfrac{1}{\omega - \omega_c}$ term can be ignored because it is analytic around the neighbourhood of $\omega = -\omega_c$. By Residue Theorem ,

$$\begin{aligned}\int_{\gamma_1} \left(\frac{1}{\omega + \omega_c} \right) \mathrm{d}\omega &= 2\pi i \times W(\gamma_1, -\omega_c) \mathrm{Res}\left(\frac{1}{\omega + \omega_c}, -\omega_c \right) \\ &= 2\pi i \times \frac{1}{2} \times 1 = \pi i,\end{aligned}$$

where $W(\gamma_1, -\omega_c)$ is the winding number of γ_1 around pole $\omega = -\omega_c$ and $\mathrm{Res}(f(z), z_k)$ is the residue of $f(z)$ at z_k. Similarly we can calculate the third integral.

So finally the value of B at frequency ω_c is,

$$B_c = \frac{2\omega_c}{\pi} \int_0^{+\infty} \frac{A - A_c}{\omega^2 - \omega_c^2} \mathrm{d}\omega. \tag{4}$$

Equation (4) has an alternative form if logarithmic scale is chosen. Suppose we define $u = \ln(\omega/\omega_c)$, we rewrite B_c as,

$$\begin{aligned}B_c &= \frac{2}{\pi} \int_0^{+\infty} \frac{A - A_c}{\omega/\omega_c - \omega_c/\omega} \frac{\mathrm{d}\omega}{\omega} \\ &= \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{A - A_c}{e^u - e^{-u}} \mathrm{d}u \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{A - A_c}{\sinh u} \mathrm{d}u.\end{aligned}$$

Bode showed with heavy algebra by integrating by parts, the above equation can be also expressed as,

$$B_c = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}A}{\mathrm{d}u} \ln \left(\coth \frac{|u|}{2} \right) \mathrm{d}u. \tag{5}$$

Notice that $\ln \left(\coth \dfrac{|u|}{2} \right) = \ln \left| \dfrac{\omega + \omega_c}{\omega - \omega_c} \right|$. If we compare Equation (5) with Equation (1), they are essentially the same except that A and B are real and imaginary components of a complex valued function $\theta(\omega)$. To see they are equivalent, let's do a simple transformation on $G(j\omega)$. Since $G(j\omega)$ is a complex number, we rewrite it according to Euler formula,

$$\begin{aligned} G(j\omega) &= |G(j\omega)|\, e^{j\angle G(j\omega)} \\ \Rightarrow \underbrace{\ln\left(G(j\omega)\right)}_{\theta(\omega)} &= \underbrace{\ln|G(j\omega)|}_{A} + j\underbrace{\angle G(j\omega)}_{B}. \end{aligned}$$

Hence it completes the proof. $\square$

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