Runge-Kutta Method Revisited

- The Euler's method is the first order Runge-Kutta method.
- The Midpoint method is the second order Runge-Kutta method.
- The fourth order Runge-Kutta method, as well as Bulirsch-Stoer method, is one of the two prevailing methods used in integration of ODEs (Ordinary Differential Equations).

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- **Defining an ODE** An ordinary differential equation can be defined as

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x, y(x)), \tag{1}$$

where $f(\cdot,\cdot)$ is a function of both x and $y(x)$, and $y(x)$ can be generalised to vector form $\boldsymbol{y}(x) = (y_1(x), y_2(x), \ldots, y_i(x), \ldots)$. Correspondingly, the Left Hand Side (LHS) of Equation (1) will be generalised to $\frac{\mathrm{d}y_i}{\mathrm{d}x}(x)$ etc., and a higher order ODE can be reduced to a *system* of first order ODEs.

- **The Euler's method** integrates the LHS of Equation (1) once, using the current point x_n and *slope* at the current point, i.e., $f(x_n, y_n)$[1], to compute the value of $y(x)$ at the next point x_{n+1}, denoted as y_{n+1}.

$$y_{n+1} = y_n + h\, f(x_n, y_n), \tag{2}$$

where $h = x_{n+1} - x_n$ is called *step size* of integration. The difference between the true value $y(x_{n+1})$ and the computed value y_{n+1} is called *local truncation error* (LTE). According to the Taylor expansion of $y(x)$ around x_n,

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \mathcal{O}(h^2),$$

where the big-$\mathcal{O}$ notation means Higher Order Terms (H.O.T., for two functions f, g that can be expressed as $f = \mathcal{O}(g)$, it means there is some positive constant C such that $|f| \leqslant C|g|$, where $|\cdot|$ is the *size* of the function.). So the LTE for the Euler's method is $\mathcal{O}(h^2)$, thus the Euler's method is first order.

[1] (x_i, y_i)'s are always interpreted as *data* points.

- **The Midpoint method** improves the Euler's method by using *average* slope $f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_1)$ at the midpoint $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_1)$, where $\Delta_1 = h\, f(x_n, y_n)$ is the increment in y by Euler's method. So by the Midpoint method, the computed y_{n+1} is

$$y_{n+1} = y_n + \Delta_2 + \mathcal{O}(h^3), \tag{3}$$

where $\Delta_2 = h\, f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_1)$ is the increment in y by Euler's method but using the average slope at middle point. Notice in Equation (3), $\mathcal{O}(h^3)$ follows the rigorous numerical analysis, which says Midpoint method raises the order of this method to two.

- **The Runge-Kutta fourth order method** further generalises the idea of Midpoint method such that the Right Hand Side (RHS) of Equation (1) f will be computed four times.

$$\begin{aligned}
\Delta_1 &= h\, f(x_n, y_n), \\
\Delta_2 &= h\, f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_1), \\
\Delta_3 &= h\, f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_2), \\
\Delta_4 &= h\, f(x_n + h, y_n + \Delta_3), \\
y_{n+1} &= y_n + \frac{1}{6}\Delta_1 + \frac{1}{3}\Delta_2 + \frac{1}{3}\Delta_3 + \frac{1}{6}\Delta_4 + \mathcal{O}(h^5).
\end{aligned} \tag{4}$$

The last equation in Equations (4) gives the average value of computed y_{n+1}.

Table 1: Interpretation of Δ_i's in Runge-Kutta fourth order formula

Increments	Interpretation
Δ_1	Increment in y by Euler's method with slope at beginning data point (x_n, y_n)
Δ_2	Increment in y by Euler's method with slope at middle data point $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_1)$
Δ_3	Repeated calculation of increment in y by Euler's method with improved slope at midpoint $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}\Delta_2)$
Δ_4	Increment in y by Euler's method with slope at ending data point $(x_n + h, y_n + \Delta_3)$

Question: Why the coefficients of the *weights* in the last equation in Equations (4) are chosen as $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{6}$? (Answer: See the derivation on the Wikipedia[2].)

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[2]See "Derivation of the Runge-Kutta fourth-order method" on `wikipedia.org`.